Dynamic Chromatic Number of Regular Graphs

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Abstract

A dynamic coloring of a graph G is a proper coloring such that for every vertex $v \in V(G)$ of degree at least 2, the neighbors of v receive at least 2 colors. It was conjectured [B. Montgomery. *Dynamic coloring of graphs*. PhD thesis, West Virginia University, 2001.] that if G is a k-regular graph, then $\chi_2(G) - \chi(G) \leq 2$. In this paper, we prove that if G is a k-regular graph with $\chi(G) \geq 4$, then $\chi_2(G) \leq \chi(G) + \alpha(G^2)$. It confirms the conjecture for all regular graph G with diameter at most 2 and $\chi(G) \geq 4$. In fact, it shows that $\chi_2(G) - \chi(G) \leq 1$ provided that G has diameter at most 2 and $\chi(G) \geq 4$. Moreover, we show that for any k-regular graph G, $\chi_2(G) - \chi(G) \leq 6 \ln k + 2$. Also, we show that for any n there exists a regular graph G whose chromatic number is n and $\chi_2(G) - \chi(G) \geq 1$. This result gives a negative answer to a conjecture of [A. Ahadi, S. Akbari, A. Dehghan, and M. Ghanbari. On the difference between chromatic number and dynamic chromatic number of graphs. *Discrete Math.*, In press].

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1 Introduction

Let H be a hypergraph. The vertex set and the hyperedge set of H are mentioned as $V(H)$ and $E(H)$, respectively. The maximum degree and the minimum degree of H are denoted by $\Delta(H)$ and $\delta(H)$, respectively. For an integer $l \geq 1$, denote by $[l]$, the set $\{1, 2, \ldots, l\}$. A proper l-coloring of a hypergraph H is a function $c : V(H) \longrightarrow [l]$ in which there is no monochromatic hyperedge in H. We say a hypergraph H is t-colorable if there is a proper t-coloring of it. For a hypergraph H, the smallest integer l so that H is l-colorable is called the chromatic number of H and denoted by $\chi(H)$. Note that a graph G is a hypergraph such that the cardinality of each $e \in E(G)$ is 2.

A proper vertex l-coloring of a graph G is called a dynamic l-coloring [14] if for every vertex u of degree at least 2, there are at least two different colors appearing in the neighborhood of v. The smallest integer l so that there is a dynamic l-coloring of G is called *the dynamic chromatic number of* G and denoted by $\chi_2(G)$. Obviously, $\chi(G) \leq \chi_2(G)$. Some properties of dynamic coloring were studied in [3, 6, 10, 11, 14, 13]. It was proved in [11] that for a connected graph G if $\Delta \leq 3$, then $\chi_2(G) \leq 4$ unless $G = C_5$, in which case $\chi_2(C_5) = 5$ and if $\Delta \geq 4$, then $\chi_2(G) \leq \Delta + 1$. It was shown in [14] that the difference between chromatic number and dynamic chromatic number can be arbitrarily large. However, it was conjectured that for regular graphs the difference is at most 2.

Conjecture 1. [14] *For any regular graph* G*,* $\chi_2(G) - \chi(G) \leq 2$

Also, it was proved in [14] that if G is a bipartite k-regular graph, $k \geq 3$ and $n < 2^k$, then $\chi_2(G) \leq 4$. This result was extended to all regular bipartite graphs in [6].

In a graph G, a set $T \subseteq V(G)$ is called a *total dominating set* in G if for every vertex $v \in V(G)$, there is at least one vertex $u \in T$ adjacent to v. The set $T \subsetneq V(G)$ is called a *double total dominating set* if T and its complement $V(G) \setminus T$ are both total dominating [6]. Also, by $\mathcal{I}(G)$ and $\mathcal{IM}(G)$ we refer to the set of independent and maximal independent sets in G, respectively.

2 results

The 2-colorability of hypergraphs has been studied in the literature and has lots of applications in the other area of combinatorics.

Theorem 1. [12] *Let H be a hypergraph in which every hyperedge contains at least k points and meets at most d other hyperedges. If $e(d+2) \leq 2^k$, then H is 2-colorable.*

Assume that G is a graph. Let $T \subseteq V(G)$ and define a hypergraph $H_G(T)$ whose vertex set is $\bigcup_{v \in T} N(v)$ and its hyperedge set is defined as follows

$$E\left(H_G(T)\right) \stackrel{\text{def}}{=} \left\{N(v) | v \in T\right\}.$$

Clearly, for any $f \in E(H_G(T))$, $\delta(G) \leq |f| \leq \Delta(G)$ and $\Delta(H_G(T)) \leq \Delta(G)$. Therefore f meets at most $\Delta(G)(\Delta(G)-1)$ other hyperedges.

It was shown by Thomassen [15] that for any k-uniform and k-regular hypergraph H, if $k \geq 4$, then H is 2-colorable. This result can be easily extended to all k-uniform hypergraphs with the maximum degree at most k [6], i.e., any k-uniform hypergraph H with $k \geq 4$ and the maximum degree at most k, is 2-colorable. By considering Theorem 1, if $e(\Delta^2(G) - \Delta(G) + 2) \leq 2^{\delta(G)}$ (in the k-regular case, $k \geq 4$), then $H_G(T)$ is 2-colorable.

Next lemma is proved in [4] and extended to circular coloring in [5].

Lemma 1. [4] *Let G be a connected graph, and let c be a $\chi(G)$-coloring of G. Moreover, assume that H is a nonempty subgraph of G. Then there exits a $\chi(G)$-coloring f of G such that:*

a) if $v \in V(H)$, then $f(v) = c(v)$ and

b) for every vertex $v \in V(G) \setminus V(H)$ there is a path $v_0 v_1 \ldots v_m$ such that $v_0 = v$, v_m is in H, and $f(v_{i+1}) - f(v_i) = 1 \pmod{\chi(G)}$ for $i = 0, 1, \ldots, m-1$.

Assume that G is a given graph. The graph G^2 is a graph with the vertex set $V(G)$ and two different vertices u and v are adjacent in G^2 if $d_G(u, v) \leq 2$, i.e., there is a walk with length at most two between u and v in G.

The connection between the independence number and the dynamic chromatic number of graphs has been studied in [1]. The first part of the next theorem improves a similar result in [1] and in the second part of it we present an upper bound for the dynamic chromatic number of graph G in terms of chromatic number of G and the independence number of G^2.

Theorem 2.

1) *For any graph G with $\chi(G) \geq 4$, $\chi_2(G) \leq \chi(G) + \alpha(G)$.*

2) *If G is a k-regular graph with $\chi(G) \geq 4$, then $\chi_2(G) \leq \chi(G) + \alpha(G^2)$.*

Proof. Let uv be an edge of G. Assume that c is a $\chi(G)$-coloring of G such that $c(u) = 1$ and $c(v) = 3$. Let H be the subgraph induced on the edge uv and f be a coloring as in Lemma 1. According to Lemma 1, $f(u) = 1$ and $f(v) = 3$. We call a vertex $v \in V(G)$, a *bad* vertex if $\deg(v) \geq 2$ and all vertices in $N(v)$ have the same color in f. Let S be the set of all bad vertices. We claim that S is an independent set in G. Suppose therefore (reductio ad absurdum) that this is not the case. We consider four different cases.

1. $\{u, v\} \subset S$. Since u and v are both bad vertices and $uv \in E(G)$, all the vertices in $N(u)$ have the color 3 and all the vertices in $N(v)$ have the color 1. Note that $\chi(G) \geq 4$. Therefore the coloring f does not satisfy Lemma 1 and it is a contradiction.

2. There exists a vertex $x \neq v$ such that $\{u, x\} \subseteq S$ and $ux \in E(G)$. Since $v \in N(u)$ and $u \in N(x)$ and both u and x are bad, all the vertices in $N(u)$ and $N(x)$ have the colors 3 and 1 in the coloring f, respectively. According to Lemma 1, there is a path $v_0 v_1 \dots v_m$ in G such that $v_0 = x$, $v_m \in V(H)$, and $f(v_{i+1}) - f(v_i) = 1 \pmod{\chi(G)}$ for $i = 0, 1, \dots, m-1$. But, this is not possible, because all the neighbors of x have the color 1 and x has the color 3.

3. There exists a vertex $y \neq u$ such that $\{v, y\} \subseteq S$ and $vy \in E(G)$. This case is the same as the previous case.

4. There are two vertices x and y in $S \setminus \{u, v\}$ such that $xy \in E(G)$. Note that according to Lemma 1, there should be at least two vertices $z \in N(x)$ and $z' \in N(y)$ such that $f(z) = f(x)+1$ and $f(z') = f(y)+1$ (modχ). Since x and y are both bad vertices, all the vertices in $N(x)$ have the color $f(y)$ and all the vertices in $N(y)$ have the color $f(x)$. Therefore $f(z) = f(y)$ and $f(z') = f(x)$ (mod $\chi(G)$), but this is not possible because $\chi(G) \geq 4$.

Now, we know S is an independent set in G and so $|S| \leq \alpha(G)$. For any vertex $w \in S$, choose a vertex $x(w) \in N(w)$ and put all these vertices in S'. Assume that $S' = \{x_1, x_2, \dots, x_t\}$. Consider a coloring f' such that f and f' are the same on $V(G) \setminus S'$ and for any $x_i \in S'$, x_i is colored with $\chi(G) + i$. One can easily check that f' is a dynamic coloring of G used at most $\chi(G) + \alpha(G)$ colors.

To prove the second part, assume that G is a k-regular graph with $\chi(G) \geq 4$. Consider the coloring f and the set S as in the previous part. Assume that $G^2[S]$, i.e., the induced subgraph of G^2 on the vertices in S, has the components $G_1^2, G_2^2, \dots, G_n^2$. Note that two different vertices $x, y \in S$ are adjacent in G^2 if and only if $N_G(x) \cap N_G(y) \neq \varnothing$ (since S is an independent set, $xy \notin E(G)$). Therefore for any $1 \leq i \leq n$, all the vertices in

$$N_i = \bigcup_{x \in V(G_i^2)} N_G(x)$$

have the same color in the coloring f. For any $1 \le i \le n$, let H_i be a hypergraph with the vertex set N_i and with the hyperedge set

$$E(H_i) = \{N(x) |\ x \in V(G_i^2)\}.$$

It is clear that H_i is a k-uniform hypergraph with $\Delta(H_i) \le k$. Since $\chi(G) \ge 4$, we have $k \ge 4$ or $G = K_4$. If $G = K_4$, then $\chi_2(G) = 4 \le \chi(G) + \alpha(G^2)$ and there is nothing to prove. Now, we can assume that $k \ge 4$. According to the discussion after Theorem 1, H_i is 2-colorable. For any $1 \le i \le n$, let (X_i^1, X_i^2) be a 2-coloring of H_i. Define f'' to be a coloring of G such that f'' and f are the same on $V(G) \setminus (\bigcup X_i^1)$ and for each $1 \le i \le n$, f'' has the constant value $i + \chi(G)$ on the vertices of X_i^1. It is easy to see that f'' is a $(\chi(G) + n)$-dynamic coloring of G. Obviously, $n \le \alpha(G^2)$ and the proof is completed. ∎

In the proof of the second part of Theorem 2, we need the 2-colorability of all H_i's and if some assumptions cause this property, then the remain of proof still works. Consequently, in view of the discussion after Theorem 1, we have the next corollary.

Corollary 1. *Let G be a graph such that $\chi(G) \ge 4$ and $e(\Delta^2(G) - \Delta(G) + 1) \le 2^{\delta(G)}$. Then $\chi_2(G) \le \chi(G) + \alpha(G^2)$.*

Remark. Note that in the proof of Theorem 2, it is shown that for any k-regular graph G with $\chi(G) \ge 4$, $\chi_2(G) \le \chi(G) + \mathrm{com}(G^2[S])$ where $\mathrm{com}(G^2[S])$ is the number of connected components of $G^2[S]$ and S is an independent set given in the proof of Theorem 2. Therefore for any graph G with $\chi(G) \ge 4$ and $e(\Delta^2(G) - \Delta(G) + 1) \le 2^{\delta(G)}$ (in k-regular case $k \ge 4$),

$$\chi_2(G) \le \chi(G) + \max_{I \in \mathcal{I}(G)} \mathrm{com}(G^2[I]).$$

It is shown in [2] that if G is a strongly regular graph except C_5 and $K_{m,m}$, then $\chi_2(G) - \chi(G) \le 1$. Note that for a graph G with diameter 2, the graph G^2 is a complete graph and $\alpha(G^2) = 1$. Therefore the second part of Theorem 2 extends this result to a larger family of regular graphs. In fact, every strongly regular graph has diameter at most 2, but according to the second part of Theorem 2, if G is a k-regular graph with diameter 2 and $\chi(G) \ge 4$, then $\chi_2(G) - \chi(G) \le 1$. Moreover, by the previous corollary, if G is a graph with diameter 2, $\chi(G) \ge 4$ and $e(\Delta^2(G) - \Delta(G) + 1) \le 2^{\delta(G)}$, then $\chi_2(G) - \chi(G) \le 1$. We restate this result in the next corollary.

Corollary 2. *Let G be a graph with diameter 2, $\chi(G) \ge 4$ and $e(\Delta^2(G) - \Delta(G) + 1) \le 2^{\delta(G)}$ (in k-regular case $k \ge 4$). Then $\chi_2(G) - \chi(G) \le 1$*

The proof of the second part of Theorem 2 strongly depended on the assumption that $\chi(G) \ge 4$. In fact the only bipartite regular graphs with diameter 2 are complete regular bipartite graphs whose chromatic number and dynamic chromatic number are 2 and 4, respectively. But $K_{m,m}^2$ is a complete graph and so $\chi(K_{m,m}) + \alpha(K_{m,m}) = 3 < \chi_2(K_{m,m})$. For the case of $\chi(G) = 3$, if we set $G = C_5$, then $C_5^2 = K_5$ and $\chi_2(C_5) = 5 > \chi(C_5) + \alpha(C_5^2)$.

Note that in the proof of Theorem 2, we assumed that $\chi(G) \geq 4$ because we want to use Lemma 1 to obtain a coloring f such that all the bad vertices related to f form an independent set in G. However, if one finds a t-coloring f of G such that the set of bad vertices related to f, S, is an independent set in G, then $\chi_2(G) \leq t+\alpha(G)$ and if G is a k-regular graph with $k \geq 4$, then $\chi_2(G) \leq t + \mathrm{com}(G^2[S])$.

Now, let G be a graph such that $e(\Delta^2(G) - \Delta(G) + 1) \leq 2^{\delta(G)}$ (in k-regular case, $k \geq 4$) and let I be an arbitrary maximal independent set in G. Consider an optimum t-coloring c of G such that I is a color class in this coloring(t is the least possible number). Define H to be a hypergraph with vertex set I and the hyperedge set $E(H) = \{N(v)|\ v \in V(G)\ \&\ N(v) \subseteq I\}$. Since $e(\Delta^2(G) - \Delta(G) + 1) \leq 2^{\delta(G)}$ (in k-regular case, $k \geq 4$), H is 2-colorable. Let (X, Y) be a 2-coloring of H. Recolor the vertices in Y with a new color $t+1$ to obtain a $(t+1)$-coloring f of G. It is readily seen that S, the set of the bad vertices related to f, is a subset of I and therefore it is an independent set. By the same argument as in the proof of the second part of Theorem 2, one can show that $\chi_2(G) \leq t + 1 + \mathrm{com}(G^2[S])$. Now, note that $t \leq \chi(G) + 1$ and so $\chi_2(G) \leq t + 1 + \mathrm{com}(G^2[S]) \leq \chi(G) + 2 + \max\limits_{P \subseteq I} \mathrm{com}(G^2[P])$. Let $\mathcal{IM}(G)$ be the set of all maximal independent sets in G. Since I is an arbitrary maximal independents set in G,

$$\chi_2(G) \leq \chi(G) + \min_{I \in \mathcal{IM}(G)} \max_{P \subseteq I} \mathrm{com}(G^2[P]) + 2.$$

In Theorem 2, we have the assumption $\chi(G) \geq 4$ and for a graph G with $\chi(G) < 4$, we can not use this theorem. In view of the above discussion, if we consider c as a $\chi(G)$-coloring of G such that the color class V_1 (all the vertices with color 1) is a maximal independent set in $G(t = \chi(G))$, then $\chi_2(G) \leq \chi(G) + \alpha(G) + 1$ and also, we have the next corollary.

Corollary 3. *Let G be a graph such that $e(\Delta^2(G) - \Delta(G) + 1) \leq 2^{\delta(G)}$ (in k-regular case, $k \geq 4$). Then $\chi_2(G) \leq \chi(G) + \alpha(G^2) + 1$.*

Erdős and Lovász [8] proved a very powerful lemma, known as the Lovász Local Lemma.

The Lovász Local Lemma.[7] Let $A_1, A_2, \ldots, A_n$ be evens in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all A_j but at most d of the other events and $\Pr(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(d+1) \leq 1$ then $\Pr(\bigcap\limits_{i=1}^{n} \bar{A}_i) > 0$

It was proved in [6] that for any k-regular graph G, the difference between dynamic chromatic number and chromatic number of G is at most $14.06 \ln k + 1$. In the next theorem we shall improve this result.

Theorem 3. *For any k-regular graph G, $\chi_2(G) - \chi(G) \leq 6 \ln k + 2$.*

Proof. It is proved in [6], that for any regular graph G, $\chi_2(G) \leq 2\chi(G)$. Therefore for any k-regular graph G with $k \leq 3$, $\chi_2(G) \leq 6 \leq 6 \ln k + 2$. Now, we can assume that $k \geq 4$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. For any permutation (total ordering) $\sigma \in S_{V(G)}$, set

$$I_\sigma = \{v \in V(G)|\ v \prec_\sigma u \text{ for all } u \in N(v)\}.$$

It is readily seen that I_σ is an independent set of G. Assume that $U \subseteq V(G)$ consists of all vertices that are not lied in any triangle. Now, choose l permutations $\sigma_1, \ldots, \sigma_l$, randomly and independently. For any $u \in U$, let A_u be the event that there are not a vertex $v \in N(u)$ and σ_i such that the vertex v precedes all of its neighbors in the permutation σ_i, i.e., $N(u) \bigcap \bigcup_{i=1}^{l} I_{\sigma_i} = \varnothing$. Since u dose not appear in any triangle, one can easily see that $\Pr(A_u) = (1 - \frac{1}{k})^{kl} \leq e^{-l}$. Note that A_u is mutually independent of all events A_v for which $d(u,v) > 3$. Consequently, A_u is mutually independent of all but at most $k^3 - k^2 + k + 1$ events. In view of Lovász Local Lemma, if $e(k^3 - k^2 + k + 2)e^{-l} \leq 1$, then none of the events A_u happens with positive probability. In other words, for $l = \lceil 3 \ln k + 1 \rceil$, there are permutations $\sigma_1, \ldots, \sigma_l$ such that A_u does not happen for any $u \in U$. It means that for any vertex $u \in U$, there is an i $(1 \leq i \leq l)$ such that $N(u) \cap I_{\sigma_i} \neq \varnothing$. Note that if we set $T = \bigcup_{i=1}^{l} I_{\sigma_i}$, then $G[T]$, the induced subgraph on T, has the chromatic number at most l. Assume that c_1 is a proper $\chi(G[T])$-coloring of $G[T]$. Now, let H be a hypergraph with the vertex set T and the hyperedge set defined as follows

$$E(H) = \{N(v) | v \in V(G),\ N(v) \subseteq T\}.$$

One can check that H is 2-colorable. If H is an empty hypergraph, there is noting to prove. Otherwise, since H is a k-uniform hypergraph with maximum degree at most k $(k \geq 4)$, H is 2-colorable. Assume that c_2 is a 2-coloring of H. It is obvious that $c = (c_1, c_2)$ is a $2l$-coloring of $G[T]$. Now, consider a $(\chi(G) + 2l)$-coloring f for G such that the restriction of f on T is the same as c. One can check that f is a dynamic coloring of G. ■

It is proved in [6] that for any $c > 6$, there is a threshold $n(c)$ such that if G is a k-regular graph with $k \geq n(c)$ then, G has a total dominating set inducing a graph with maximum degree at most $2c \log k$ (for instance if we set $c = 7.03$ then $n(c) \leq 139$). Note that in the proof of previous theorem, it is proved that any triangle free k-regular graph G has a total dominating set T such that the induced subgraph $G[T]$ has the chromatic number at most $\lceil 3 \ln k + 1 \rceil$.

In the rest of the paper by $G \times H$ and $G \square H$, we refer to the Categorical product and Cartesian product of graphs G and H, respectively. It is well-known that if G is a graph with $\chi(G) > n$, then $G \times K_n$ is a uniquely n-colorable graph, see [9].

It was conjectured in [1] that for any regular graph G with $\chi(G) \geq 4$, the chromatic number and the dynamic chromatic number are the same. Here, we present a counterexample for this conjecture.

Proposition 1 *For any integer $n > 1$, there are regular graphs with chromatic number n whose dynamic chromatic number is more than n.*

Proof. Assume that G_1 is a d-regular graph with $\chi(G_1) > n$ and $m = |V(G_1)|$. Set $G_2 = G_1 \square C_{(n-1)(d+2)+1}$ and $G' = G_2 \times K_n$. Note that G' is a uniquely n-colorable graph with regularity $(n-1)(d+2)$. Consider the n-coloring $(V_1, V_2, \ldots, V_n)$ for

G', where for $1 \leq i \leq n$, $V_i = \{(g,i) \mid g \in V(G_2)\}$. It is obvious that $|V_i| = m((n-1)(d+2)+1))$ is divisible by $(n-1)(d+2)+1$. Now, for each $1 \leq i \leq n$, consider $(S_1^i, S_2^i, \ldots, S_m^i)$ as a partition of V_i such that $|S_j^i| = (n-1)(d+2)+1$. Now, for any $1 \leq i \leq n$ and $1 \leq j \leq m$, add a new vertex s_{ij} and join this vertex to all the vertices in S_j^i to construct the graph G. Note that G is an $((n-1)(d+2)+1)$-regular graph with $\chi(G) = n$. Now, we claim that $\chi_2(G) > n$. To see this, assume that c is an n-dynamic coloring of G. Since G' is uniquely n-colorable, the restriction of c to $V(G)$ is $(V_1, V_2, \ldots, V_n)$. In the other words, all the vertices in V_1 have the same colore in c. But, all the neighbors of s_1^1 are in V_1 and this means that c is not a dynamic coloring. ■

However, it can be interesting to find some regular graph G with $\chi(G) \geq 4$ and $\chi_2(G) - \chi(G) \geq 2$.

Also, as a generalization of Conjecture 1, it was conjectured [1] that for any graph G, $\chi_2(G) - \chi(G) \leq \lceil \frac{\Delta(G)}{\delta(G)} \rceil + 1$. Here, we give a negative answer to this conjecture. To see this, assume that G_1 is a graph with $\chi(G_1) \geq 3$ and n vertices such that $n > 3\Delta(G_1) + 5$. For any 2-subset $\{u, v\} \subseteq V(G_1)$, add a new vertex x_{uv} and join this vertex to the vertices u and v. Let G be the resulting graph form G_1 by using this construction. Note that $\chi_2(G) \geq n$, $\Delta(G) = \Delta(G_1) + n - 1$, $\delta(G) = 2$ and $\chi(G) = \chi(G_1)$. Therefore if the conjecture was true, then we would have

$$n - \Delta(G_1) - 1 \leq n - \chi(G_1) \leq \chi_2(G) - \chi(G) \leq \lceil \frac{\Delta(G_1) + n - 1}{2} \rceil + 1.$$

Note that it is not possible because $n > 3\Delta(G_1) + 5$, and consequently, $n - \Delta(G_1) - 1 > \lceil \frac{\Delta(G_1)+n-1}{2} \rceil + 1$.

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